An energy model for event-based control of a switched integrator

Simona Mihăiţă * **Stéphane Mocanu** *

* *Control system department, GIPSA-lab, UMR 5216, Grenoble-INP, France.*

Abstract: In this paper we present a method for constructing an approximative stationary energy model of a stochastic switching system while applying event-based control. The main model used is a multi-state integrator with random switching behavior, upon event occurrence, which has been described using a continuous Markov chain. Applying event-based control has been done in the purpose of maintaining the continuous system state variable between extreme boundaries. The sojourn times and the probabilities to hit the limits have also been used for the computation of the average energy consumed during the stationary mode of the Markov chain. Mean times between intermediary stopping points and extreme limits are also provided as well as a quadratic criterion for square deviations from imposed target limits.

Keywords: Stochastic switching systems, event-based control, energy modeling

1. INTRODUCTION

Automatic control for stochastic switching systems has come to play an important role in significant branches of science and industry, especially in the production planning for manufacturing systems. Stochastic switching systems have been used to model a lot of practical dynamical systems, as they are described by the interaction between continuous and discrete dynamics (Liberzon [2003]). Many examples can be found in transportation systems (Pola et al. [2003], Varaiya [1993]), robotics (Egerstedt [2000]), communication networks (Hespanha [2005]), automotive systems and even in biological system modeling (Khare et al. [2005]). Many viewpoints have been developed, and although researchers focused on the study of the system discrete behavior, simulation problems have appeared. On the other hand, the automatic control researchers focus on the continuous aspect of the systems with discrete switching, but still debate problems such as stability analysis and control synthesis. The continuous dynamic of a stochastic switching system can be modeled using differential equations (Branicky [1998]):

$$\dot{x}(t) = \xi(t), t \geq 0.$$

where $x(t)$ is the continuous state component taking real values, $\xi(t)$ is a vector field that depends in general on $x(t)$ and on the discrete behavior of the system. Stochastic switching systems or *jump systems* have been frequently used for modeling practical systems with abrupt changes in their behavior which may experience breakdown of components, repair or abrupt environmental disturbances, and their continuous behavior can be often associated with a Markov process.

On the other hand, the discrete behavior of a jump system can also be influenced by random events which will change the system evolution. A special control is needed upon event arrival, which is called event-based control. This requirement is wide spread in many technological areas, such as transportation networks, health care, energy efficient operation of vehicles and manufacturing. Event-based control has become an attractive approach to solve control problems for systems with certain rate limitations as in process industry (Pettersson et al. [2006], Guzzella and Onder [2006]) or with expensive control actions like in communication networks (Cogill et al. [2007], Jacobson [1988], Kelley [1985]). The control action is only applied when is needed, reducing the rate when the system must be actuated; it has also been the standard form of control used in biological systems as well (Wilson [1999]). There is little theory which offers details about designing an event-based controller and estimation strategies (Kofman and Braslavsky [2006], Rabi and Baras [2007], Tabuada [2007], Åström [2002]), therefore the control problem can be often seen as a Markov decision process. But the optimal control function for such a process can become very complicate, especially when the state number increases.

In this paper we develop a preliminary approximative model for the stationary energy consumed when event-based control is applied to a stochastic switching system. The control will be applied over a multi-state integrator system bounded by upper and lower limits. The uncontrolled and controlled system definitions will be provided in Section 2. Section 3 presents the stationary energy model which have been built using the mean hitting times to reach the limits, as well as the probabilities to hit the upper or the lower limit. Section 4 provides a quadratic criterion which is used for computing the total cost associated to the square deviation from the target level of the continuous state variable.

2. BASE MODEL

The base model we have considered for our study is a particular type of a stochastic switching system, in which the switching between the states does not depend on the

evolution of the continuous state component. Section 2.1 presents the definition of the system while in section 2.2 we present the control model of the system.

2.1 Integrator definition

We will first consider a simple multi-variable integrator system which can be described by the following equation:

$$\begin{cases} \dot{x}(t) = r_{Z(t)} \\ x(0) = x_0 \end{cases} \quad (1)$$

where $Z(t)$ is the continuous-time Markov chain describing the system mode at time t and taking values in the finite state space $S = \{1, 2, .., N\}$, $x(t) \in \Re$ is the state variable, $x_0 \in \Re$ is the initial state of the system and $r_i, i \in \{1, 2, ..N\}$ are constants (which can be seen as state rewards) so that:

$$\begin{cases} r_i > 0, & \forall i \in \{1, 2, .., k\} \\ r_j < 0, & \forall j \in \{k+1, ..N\} \end{cases} \quad (2)$$

For simplicity we assume that all r_i are non-zero.

The continuous behavior of the system is described by $x(t)$, while the discrete aspect is characterized by the system random transitions between the states. The associated continuous-time Markov chain is characterized by the following transition rate matrix:

$$\mathbf{Q} = \begin{pmatrix} -\sum_{j \neq 1} \lambda_{1j} & \lambda_{12} & .. & \lambda_{1N} \\ \lambda_{21} & -\sum_{j \neq 2} \lambda_{2j} & .. & \lambda_{2N} \\ .. & .. & .. & .. \\ \lambda_{N1} & \lambda_{N2} & .. & -\sum_{j \neq N} \lambda_{Nj} \end{pmatrix}$$

where λ_{ij} is the transition rate from state i to state j. The stationary state probability vector of the Markov chain is:

$$\pi = [\pi_0, \pi_1, ...]$$

which satisfies the stationary condition: $\pi Q = 0$. As the system switches randomly between the states, an adapted event-based control will be applied each time certain conditions are met.

2.2 Controlled Integrator

We adopt a simple event-driven controller. Previous works (mainly Åström [2002]) showed that in some cases event-based control can provide better performances than sampled data control. On the other hand event-based control arise quite naturally in switched systems while "switching" is an event occurrence. It has to be proved but we suspect that the event-driven control is better adapted to switching systems than the sampled data control. An event-based controller may switch control policy on plant dynamics switching and then be more reactive.

The main objective for the considered integrator system (1) while being in the stationary mode is to maintain the system state variable between extreme limits: $x(t) \in (X_{min}, X_{max})$. Each time one limit is reached, event-based control will be applied until $x(t)$ reaches one of the stopping control boundaries (X_L, X_H); between these stopping boundaries no control is needed. When the upper limit has been reached $(x(t) = X_{max})$, a *high* control will be applied until $x(t)$ reaches the upper stopping control limit $(x(t) = X_H)$. Similarly, a low event-based control will be applied from the lower limit $(x(t) = X_{min})$ to the lower stopping control limit$(x(t) = X_L)$. The stochastic equation for the controlled system becomes:

$$\begin{cases} \dot{x}(t) = r_{Z(t)} + u_{Z(t)}(x(t)) \\ x(0) = x_0 \end{cases} \quad (3)$$

where

$$u_{Z(t)}(x(t)) = \begin{cases} 0 & ,\text{if } C_1 \\ -QH_i & ,\text{if } C_2, \forall i \in \{1, 2, ..k\} \\ +QL_j & ,\text{if } C_3, \forall j \in \{k+1, ..N\} \end{cases}$$

$$\begin{cases} C_1: & (x(t) \in [X_L, X_H]) \vee \\ & (x(t) \in (X_H, X_{max}) \wedge u(Z(t - \Delta t)) = 0) \vee \\ & (x(t) \in (X_{min}, X_L) \wedge u(Z(t - \Delta t)) = 0) \\ C_2: & (state = i) \wedge [(x(t) = X_{max}) \vee \\ & (x(t) \in (X_H, X_{max}) \wedge u(Z(t - \Delta t)) \neq 0)] \\ C_3: & (state = j) \wedge [(x(t) = X_{min}) \vee \\ & (x(t) \in (X_{min}, X_L) \wedge u(Z(t - \Delta t)) \neq 0)] \end{cases} \quad (4)$$

We use QH_i to denote the *high* control applied in the state i, $\forall i \in \{1, 2, ..k\}$ and QL_j for *low* control in the state j, $\forall j \in \{k+1, ..N\}$. The Δt notation represents an infinitesimal time interval. Recalling (2), in order to achieve the control objective, (QH_i, QL_j) will be applied so that:

$$\begin{cases} r_i - QH_i < 0 & ,\forall QH_i > r_i > 0 \\ r_j + QL_j > 0 & ,\forall QL_j > 0 \end{cases} \quad (5)$$

Equation (4) contains the control law to be applied which depends on the Markov system states $(S = \{1, 2, ..N\})$ and on the system state $x(t)$.

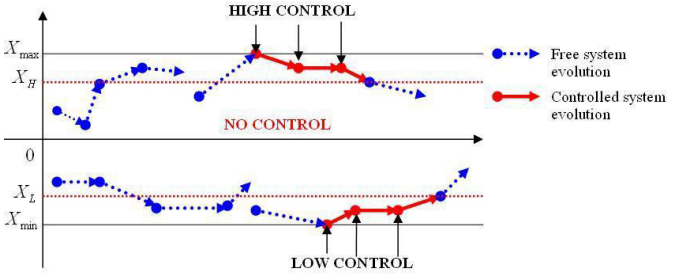

Fig. 1. Sample paths with or without control.

A graphical representation can be seen in Figure 1 which corresponds to the control law evolution described in the following:

- *no control* is applied if:
  - the state variable $(x(t))$ is in the no control area: (X_L, X_H) or
  - the state variable is in the upper control interval (X_H, X_{max}) and no control was needed before the current time t or

· the state variable is in the lower control interval (X_{min}, X_L) and no control was needed before the current time t;

- *high* control is applied if the system is in state L_i with $r_i > 0$ and the maximal limit has been reached $(x(t) = X_{max})$ or if the state vector is still in the upper control interval although upper control has been applied before t; this means we have to continue applying the control until $x(t) \leq X_H$;
- *low* control is applied if the system is in state L_j with $r_j < 0$ and the minimal limit has been reached $(x(t) = X_{min})$ or if the state vector is still in the lower control interval although lower control has been applied before t; the control has to be reapplied until $x(t) \geq X_L$.

3. STATIONARY ENERGY MODEL

In this section we built the stationary energy model for the considered stochastic switching system. Section 3.1 presents the initial conditions and problem definition, followed by Section 3.2 in which we present the approximations we have applied over the model. Section 3.3 contains the computation of the energy probabilities, while Section 3.4 presents the final energy formulas we have obtained.

3.1 Description

We are interested in the long-run model of the control energy consumption. In other words we build a model for the average energy consumed in the stationary regime of the Markov chain $Z(t)$.

Applying *low* and *high* control is an energy consuming process depending on the current state of the system and on the extreme boundaries. We have considered that the system randomly passes from *no control* area to *low* or *high* control, but will not switch directly from *low* control to *high* control or viceversa. By taking into account the control boundaries imposed in section 2.2 and the random switching between the states, the following four cases can be analyzed, as shown in Figure 2:

(1) from X_H the system evolves either to X_{max} ($\Delta_1 = X_{max} - X_H$) during T_1, where it needs *high* control to be applied until it reaches X_H (during T_2) or
(2) it can also evolve from X_H to X_{min} ($\Delta_2 = X_H - X_{min}$) during T_3, where it needs *low* control to be applied until X_L is being reached (during T_4);
(3) from X_L the system can either reach X_{min} ($\Delta_3 = X_L - X_{min}$) during T_5, where *low* control is needed until X_L is reached (during T_6) or
(4) it can hit the X_{max} limit ($\Delta_4 = X_{max} - X_L$) during T_7, where *high* control is needed until X_H is being reached (during T_8).

As it can be seen from Figure 2 the returning of the system to the control boundaries after applying the control indicates a random but "cyclic" behavior of the system in stationary mode. As we are also interested in applying the control over the system with a minimal cost, a quadratic criterion is needed in order to minimize undesired deviations from the control targets. Section (4) presents the method to obtain total cost associated to the square deviations from the target limit.

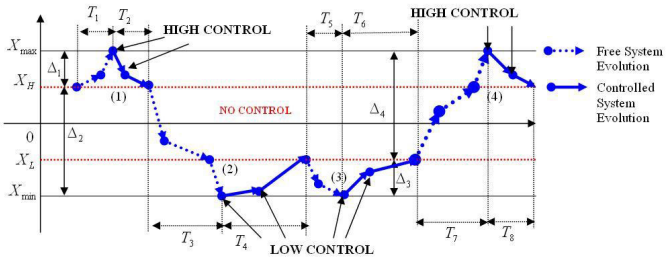

Fig. 2. Possible sample paths from control stopping limits.

In each of the possible cases presented above, specific amounts of energy are being consumed with certain probabilities $(p_j, j = 1..4)$ therefore the total possible energy consumed by the system in state i while passing between the above paths can be written as:

$$En_{tot_i} = p_1 E_{1,i} + p_2 E_{2,i} + p_3 E_{3,i} + p_4 E_{4,i}. \quad (6)$$

where $E_{j,i}, j = 1..4, i = 1..N$ is the mean energy consumed in state i when being in the one of the above cases.

An exact energy model needs knowledge of the process probabilities to hit X_{max} or X_{min} while starting in X_H, respectively X_L – the probabilities p_i – and the associated mean times to hit the respective boundaries. As it is well know in stochastic models the time to hit a boundary (exit time) cannot be computed except the case of diffusion processes. Therefore we will compute an approximation of the energy model.

3.2 Approximations

The first approximation we made was to consider only two values for the integrator constant (this an aggregation based approximation). Let the homogeneous Markov chain $Z(t)$ be in the stationary regime. Then the probability of the states equals the stationary probability vector π. Let the states of the Markov chain be ordered as in (2). We can therefore define the mean positive integrator constant and the mean negative integrator constant as :

$$v_1 = \left(\sum_{i=1}^{k} \pi_i r_i\right) / \left(\sum_{i=1}^{k} \pi_i\right) > 0, \quad (7)$$

$$v_2 = \left(\sum_{j=k+1}^{N} \pi_j r_j\right) / \left(\sum_{j=k+1}^{N} \pi_j\right) < 0. \quad (8)$$

Taking into consideration that the partition of the rewards defines a partition of the states in the Markov chain, we are interested in the sojourn time in these partition. The probability distribution of these sojourn times are easily computed using the formulas in Rubino and Sericola [1989]. Let T_+ and T_- be the random variables characterizing the sojourn time in the positive reward states, respectively the sojourn time in the negative reward state. Then the associated probability densities are given by (see Rubino and Sericola [1989])

$$f_{T_+}(t) = \alpha_1^T e^{G_1 t}(-G_1 \mathbf{1})$$
$$f_{T_-}(t) = \alpha_2^T e^{G_2 t}(-G_2 \mathbf{1}), \quad (9)$$

where G_1 and G_2 are the corresponding sub-generators (sub-matrices of Q) characterizing the transitions inside the respective subsets of states, α_1 and α_2 are probability vectors given by (Rubino and Sericola [1989]):

$$\boldsymbol{\alpha}_1^T = \boldsymbol{\pi}_+^T G_1/(\boldsymbol{\pi}_+^T G_1 \mathbf{1})$$
$$\boldsymbol{\alpha}_2^T = \boldsymbol{\pi}_-^T G_2/(\boldsymbol{\pi}_-^T G_2 \mathbf{1}), \quad (10)$$

with π_+ and π_- being the subvectors of π corresponding to the state partition ($\boldsymbol{\pi} = [\boldsymbol{\pi}_+, \boldsymbol{\pi}_-]$). Finally, $\mathbf{1}$ is a vector of adequate dimension whose entries equal 1. According with the definitions of the random variables T_+ and T_- one can define the random variables $X_+ = v_1 T_+$ and $X_- = -v_2 T_-$ modeling the increase, respectively the decrease of $x(t)$ during the sojourn in the positive, respectively negative, rate states. Then, the respective probability densities are:

$$f_{X_+}(t) = \boldsymbol{\alpha}_1^T e^{G_1' t}(-G_1'\mathbf{1}), \text{ where } G_1' = G_1/v_1$$
$$f_{X_-}(t) = \boldsymbol{\alpha}_2^T e^{G_2' t}(-G_2'\mathbf{1}), \text{ where } G_2' = -G_2/v_2 \quad (11)$$

The X_+ and X_- notations denote the changes of the x(t) value in the two state sets, therefore the second approximation we made was to construct the probabilities of hitting one of the boundaries using only the difference between these changes. This is an approximation while we replace, for example, the event: "the process hits the upper boundary before hitting the lower boundary" with the event: "the process hits the upper boundary".

The third approximation regards the mean time that the system needs to pass from the *high* control to the *no* control area and which depends on the difference between the upper imposed limit (X_{max}) and the upper stopping point (X_H), as well as on the control measures applied over the states of the system. The analogous applies to the mean time that the system need to pass from the *low* control area to the *no* control one. The definition and usage are provided in Section 3.4.

3.3 Mean hitting times and probabilities

Using the probability densities computed in (11) we calculate the probability of hitting either the lower or the upper bound starting from X_H or X_L. These will be the probabilities of the Markov chain to spend a sufficient time in the state-sets "+" or "-" in order to hit the respective boundaries. The first step is to compute the probability density of the net change in the value of $x(t)$, i.e. the probability densities of the difference of random variables $X_+ - X_-$ and $X_- - X_+$. According to the classical probability theory these densities are given by the cross-correlation products of the individual densities. Therefore:

$$f_{X_+ - X_-}(x) = \int_0^\infty f_{X_-}(y) f_{X_+}(x+y)dy$$
$$f_{X_- - X_+}(x) = \int_0^\infty f_{X_+}(y) f_{X_-}(x+y)dy. \quad (12)$$

Using (10-11) in (12) we obtain after some calculus:

$$f_{X_+ - X_-}(x) = \boldsymbol{\alpha}_1^T e^{G_1' x}(-G_1'\boldsymbol{K_1}),$$
$$f_{X_- - X_+}(x) = \boldsymbol{\alpha}_2^T e^{G_2' x}(-G_2'\boldsymbol{K_2}), \quad (13)$$

where

$$\boldsymbol{K_1} = \int_0^\infty e^{G_1' y}\mathbf{1}\boldsymbol{\alpha}_2^T e^{G_2' y}(-G_2'\mathbf{1})dy$$
$$\boldsymbol{K_2} = \int_0^\infty e^{G_2' y}\mathbf{1}\boldsymbol{\alpha}_1^T e^{G_1' y}(-G_1'\mathbf{1})dy.$$

It follows that the probabilities to hit either X_{max} or X_{min} conditional that the initial state is X_H are:

$$p_{X_H X_{max}} = P\{X_+ - X_- \geq \frac{\Delta_1}{v_1}\} = K_H \boldsymbol{\alpha}_1^T e^{G_1'\Delta_1}\boldsymbol{K_1},$$
$$p_{X_H X_{min}} = P\{X_- - X_+ \geq \frac{\Delta_2}{v_2}\} = K_H \boldsymbol{\alpha}_2^T e^{G_2'\Delta_2}\boldsymbol{K_2},$$

with K_H being a normalizing constant such that $p_{X_H X_{max}} + p_{X_H X_{min}} = 1$. In a similar way the probabilities to hit X_{max} or X_{min} starting from X_L are:

$$p_{X_L X_{max}} = P\{X_+ - X_- \geq \frac{\Delta_4}{v_1}\} = K_L \boldsymbol{\alpha}_1^T e^{G_1'\Delta_4}\boldsymbol{K_1},$$
$$p_{X_L X_{min}} = P\{X_- - X_+ \geq \frac{\Delta_3}{v_2}\} = K_L \boldsymbol{\alpha}_2^T e^{G_2'\Delta_3}\boldsymbol{K_2}.$$

One needs to evaluate the probabilities p_1 to p_4 used in (6). Let us consider, for example, p_1. This probability corresponds to the event: "system starts in state X_H and will hit X_{max}". Let p_H, respectively p_L be the probabilities of the events "system starts in state X_H (respectively X_L)". Then $p_1 = p_H p_{X_H X_{max}}$. But the event: "system starts in state X_H" is the same (given the control policy) with the event "last time the system has started in X_H or X_L and hit X_{max}". The probability of the later event is $p_H p_{X_H X_{max}} + p_L p_{X_L X_{max}}$. Together with $p_H + p_L = 1$ we can easily obtain:

$$p_H = \frac{p_{X_L X_{max}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}$$
$$p_L = \frac{p_{X_H X_{min}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}.$$

By similar calculations, we obtain :

$$p_1 = p_H p_{X_H X_{max}} = \frac{p_{X_L X_{max}} p_{X_H X_{max}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}$$
$$p_2 = p_H p_{X_H X_{min}} = \frac{p_{X_L X_{max}} p_{X_H X_{min}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}$$
$$p_3 = p_L p_{X_L X_{min}} = \frac{p_{X_H X_{min}} p_{X_L X_{min}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}$$
$$p_4 = p_L p_{X_L X_{max}} = \frac{p_{X_H X_{min}} p_{X_L X_{max}}}{p_{X_L X_{max}} + p_{X_H X_{min}}}.$$

The next step in order to model the behavior of the system is to compute the mean time to hit the boundaries starting from either X_H or X_L. Starting, for example, from X_H the average behavior of the system going to X_{max} is the following: during a period corresponding to the sum of the mean sojourn times in positive reward states and negative reward states, an average reward $E[X_+ - X_-]$ is accumulated (we assumed that the system will hit the X_{max} limit). Then the average hitting time of X_{max} while starting from X_H is:

$$T_1 = \frac{\Delta_1}{E[X_+ - X_-]}(E[T_+] + E[T_-]). \quad (14)$$

From (9) and (13) we obtain:

$$E[X_+ - X_-] = -\boldsymbol{\alpha}_1^T {G'}_1^{-1} \boldsymbol{K}_1.$$
$$E[T_+] = -\boldsymbol{\alpha}_1^T G_1^{-1} \mathbf{1}.$$
$$E[T_-] = -\boldsymbol{\alpha}_2^T G_2^{-1} \mathbf{1}.$$

Inserting in (14) we obtain:

$$T_1 = \frac{\Delta_1(\boldsymbol{\alpha}_1^T G_1^{-1} + \boldsymbol{\alpha}_2^T G_2^{-1})\mathbf{1}}{\boldsymbol{\alpha}_1^T {G'}_1^{-1} \boldsymbol{K}_1}.$$

In a similar way, the mean times to hit the *high* or the *low* boundary starting either X_H or X_L can be expressed as:

$$T_3 = \frac{\Delta_2(\boldsymbol{\alpha}_1^T G_1^{-1} + \boldsymbol{\alpha}_2^T G_2^{-1})\mathbf{1}}{\boldsymbol{\alpha}_2^T {G'}_2^{-1} \boldsymbol{K}_2}.$$
$$T_5 = \frac{\Delta_3(\boldsymbol{\alpha}_1^T G_1^{-1} + \boldsymbol{\alpha}_2^T G_2^{-1})\mathbf{1}}{\boldsymbol{\alpha}_2^T {G'}_2^{-1} \boldsymbol{K}_2}.$$
$$T_7 = \frac{\Delta_4(\boldsymbol{\alpha}_1^T G_1^{-1} + \boldsymbol{\alpha}_2^T G_2^{-1})\mathbf{1}}{\boldsymbol{\alpha}_1^T {G'}_1^{-1} \boldsymbol{K}_1}.$$

Obviously, these approximations are underestimating the real hitting times while (due to the reward rates aggregations but also because we ignore the dependence between certain events). However the formulas are very simple and our numerical experiments show that quite often the numerical approximation is good. These mean hitting times will be used for the energy formulas in the next section.

3.4 Energy formulas

In this section we develop formulas for the consumed control energy. Considering the case represented in Fig. 2, one can write the conditional expectations over the control time intervals (Δ_1, Δ_3) control as:

$$e_{1,i} = e_{4,i} = \int_0^{T_2} QH_i^2 dt = QH_i^2 \cdot T_2 \tag{15}$$

$$e_{2,i} = e_{3,i} = \int_0^{T_4} QL_i^2 dt = QL_i^2 \cdot T_4 \tag{16}$$

which can be used in computing the total *high* or *low* control energies (6) as:

$$EH_{tot,i} = p_1 \cdot \frac{e_{1,i}}{T_1 + T_2} + p_4 \cdot \frac{e_{4,i}}{T_7 + T_8} \tag{17}$$

$$EL_{tot,i} = p_2 \cdot \frac{e_{2,i}}{T_3 + T_4} + p_3 \cdot \frac{e_{3,i}}{T_5 + T_6} \tag{18}$$

T_2, T_4 stand for the mean times that the system needs to pass from the *high* or *low* control area to the *no* control area, and which can be expressed as:

$$T_2 = T_8 = \frac{\Delta_1}{\sum_{i=1}^N \pi_i(QH_i - r_i)} \tag{19}$$

$$T_4 = T_6 = \frac{\Delta_3}{\sum_{j=1}^N \pi_j(QL_j + r_j)} \tag{20}$$

As we have initially considered , the system has multiple states, therefore we can express the total consumed energy while switching between the N states as:

$$En_{tot} = \sum_{i=1}^{N} \pi_i(EH_{tot,i} + EL_{tot,i}). \tag{21}$$

4. QUADRATIC COST CRITERION

One can finally define the stochastic equivalent of a quadratic cost criterion. Let X_T be a target level for $x(t)$, $X_T \in (X_L, X_H)$ (one can assume, more generally that $X_T \in (X_{min}, X_{max})$, but this will just add some calculus complexity). One would also like to minimize the average quadratic deviation of $x(t)$ from X_T. Let Y denote the random variable representing this deviation. Then a quadratic cost criterion could be stated as:

$$J = qE[Y^2] + rEn_{tot}, \tag{22}$$

for some positive q and r, where $E[Y^2]$ is the second moment of Y. In the following we construct an approximation of the above second moment of Y by a similar analysis as in the previous section.

Assume that the process starts in the point X_H and there is no control. Then the random variable Y can be expressed as :

$$Y = \begin{cases} X_H - X_T + (X_+ - X_-);\ Y > X_H - X_T \\ X_H - X_T - (X_- - X_+);\ Y < X_H - X_T \end{cases}$$

While X_H and X_T are constants it follows that the probability density of Y may be constructed from $f_{X_+ - X_-}(x)$ or $f_{X_+ - X_-}(x)$ by simple translations. The continuous part of the density function of Y is :

$$f_{1Y}(y) =$$
$$\begin{cases} f_{X_- - X_+}((X_H - X_T) - y);\ y \in (X_{min} - X_T, X_H - X_T) \\ f_{X_+ - X_-}(y - (X_H - X_T));\ y \in (X_H - X_T, X_{max} - X_T) \end{cases}$$

Then the conditional second moment of Y starting in X_H is :

$$E_1[Y^2] = \int_{X_{min} - X_T}^{X_H - X_T} y^2 f_{X_- - X_+}((X_H - X_T) - y)dy + \\ + \int_{X_H - X_T}^{X_{max} - X_T} y^2 f_{X_+ - X_-}(y - (X_H - X_T))dy$$

In a similar way, if the process starts in X_L the density function of Y is :

$$f_{2Y}(y) =$$
$$\begin{cases} f_{X_- - X_+}((X_L - X_T) - y);\ y \in (X_{min} - X_T, X_L - X_T) \\ f_{X_+ - X_-}(y - (X_L - X_T));\ y \in (X_L - X_T, X_{max} - X_T) \end{cases}$$

and the conditional second moment of Y starting in X_L:

$$E_2[Y^2] = \int_{X_{min} - X_T}^{X_L - X_T} y^2 f_{X_- - X_+}((X_L - X_T) - y)dy + \\ + \int_{X_L - X_T}^{X_{max} - X_T} y^2 f_{X_+ - X_-}(y - (X_L - X_T))dy$$

Once the system has hit the upper limit X_{max} the random variable Y will take values in $(X_H - X_T, X_{max} - X_T)$ and can approximately be expressed using the determinist expression (based on the (19) approximation):

$$y_3(t) = X_{max} - X_T - \left(\sum_{i=1}^{N_H} \pi_i(QH_i - r_i)\right) t.$$

Then the associated quadratic cost is

$$E_3 = \int_0^{T_2} y_3^2(t)dt.$$

In a similar way if the system hits the lower limit X_{min} the approximate evolution of Y is

$$y_4(t) = X_{min} - X_T + \left(\sum_{j=1}^{N_L} \pi_j (QL_j + r_j) \right) t.$$

In a similar way, the associated quadratic cost is

$$E_4 = \int_0^{T_4} y_4^2(t) dt.$$

N_H and N_L denote the number of times we apply high or low control. Finally the total cost associated to the square deviation of the trajectory from the target X_T is:

$$E[Y^2] = (p_1 + p_2)E_1[Y^2] + (p_3 + p_4)E_2[Y^2] + \\ + (p1 + p_4)E_3 + (p_2 + p_3)E_4.$$

Along with the total energy formula (21) it can be used to apply the quadratic criterion. This total cost of the square deviation can be minimized by using nonlinear optimal control applied over the stochastic switching system.

5. CONCLUSIONS

In this paper we have studied the energy modeling of a multi-state integrator system characterized by stochastic switching between the states. The goal was to maintain the system state variable between imposed boundaries while applying event-based control. Due to the little theory in computing the exit time for stochastic models, an aggregated approximation energy model has been constructed. We have determined the probabilities to hit the limits depending on the sojourn times between the two state sets, the probabilities to hit the limits starting intermediary points, as well as the mean times spent by the system when passing from *no* control to *low* or *high* control and viceversa.

A possible direction for future work is to investigate the application of an optimal event-based control over the system in order to obtain a control energy minimization as well as the computation of optimal stopping points of the system.

5.1 Note on the numerical results

We compared our analytical model with the discrete event simulation results obtained for a large number of replications. The numerical accuracy is very variable. The approximation of the sojourn time in controlled states is always accurate and even very precise for larger values. The accuracy of the probabilities to hit a boundary is less good but seems to be more accurate in the cases with balanced rewards (i.e. if the absolute values of the positive and negative aggregated rewards v_1 and v_2 are close). The less accurate approximation is the hitting time value. Again, when the rewards are balanced the approximation is good but we experienced errors up to 50% in some unbalanced cases. Our further research points also to the improvement of the approximation error.

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