Probabilistic analysis of a class of continuous-time stochastic switching systems with event-driven control

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ABSTRACT. *In this paper we present a probabilistic approach for building the energy model when the event-driven control is applied over a class of continuous-time stochastic switching system. The system we study is a stochastic switching integrator which we model using Continuous-time Markov Chains. When analyzing the stochastic evolution of the system, we identify a restart periodic behavior, which leads to the analysis of four possible scenarios of evolution. We therefore propose new analytic methods to determine the parameters needed to compute the control energy, such as: the second order performability moment of the state variable, the exit times from the control zone, as well as the control periods, which are characteristic to each scenario. In the final part we validate the methods through numerical examples of two and four-state stochastic switching integrators.*

RÉSUMÉ. *Dans cet article nous présentons une méthode probabiliste pour construire le modèle énergétique quand le contrôle basé sur les événements est appliqué sur une classe des systèmes stochastiques à commutation. On étudie un intégrateur stochastique à commutation modélisé à l'aide des Chaînes de Markov en Temps Continu. En analysant l'évolution du système nous identifions un comportement périodique de redémarrage, qui nous conduit vers l'analyse de quatre scénarios d'évolution. Ce travail propose des nouvelles méthodes analytiques pour déterminer: les moments de performabilité du deuxième degré de la variable d'état, les temps de sortie de la zone de contrôle et les temps nécessaires pour appliquer le contrôle. Dans la dernière partie de l'article nous validons les méthodes proposées à travers des exemples numériques pour des systèmes stochastiques à commutation avec deux, respectivement quatre états.*

KEYWORDS: *stochastic switching systems, continuous-time Markov chains, event-driven control, backward Kolmogorov equations.*

MOTS-CLÉS : système stochastiques à commutation, chaînes de Markov en temps Continu, contrôle basé sur les événements, équations Kolmogorov arrière.

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1. Introduction

In the past few decades, stochastic switching systems (SSS) have known an increasing popularity, as they can easily model the behavior of many dynamical systems, which are subject to random failures of components, sudden environmental disturbances or changes in the system interconnection. Originally the SSS were an important class of hybrid dynamical systems, containing a continuous part, usually represented by a family of subsystems driven by differential equations, and a discrete part, which can be a logical rule for switching between these subsystems. The idea of introducing the stochastic aspect in the hybrid systems is not new, some of the first models and approaches being initially presented in (Lygeros *et al*., 2008). The application domain of such systems expanded to robotics (Egerstedt, Hu, 2002), networking (Strelec *et al*., 2012), transportation systems (Pola *et al*., 2003), automotive systems (Balluchi *et al*., 2000) and biological systems (Kumar *et al*., 2013). A recent analysis of the progress made in the area of stochastic hybrid systems can be found in (Lygeros, Prandini, 2010).

1.1. Contributions

The system we study throughout this paper is a particular type of stochastic switching systems, more specifically we use a multi-state stochastic switching integrator (SSI) which we model using continuous-time Markov Chains with finite state space (presented in Section 2). This type of systems plays a key role for example in the performance evaluation of water tanks or in telecommunication networks, as their behavior can be compared to that of fluid queues. Regarding the stationary analysis of fluid queues driven by countable state space Markov chains, in (Guillemin, Sericola, 2007) the authors propose an algorithm for computing the stationary probability distribution of the buffer level in the fluid queue.

For the SSI we consider in this paper, we propose an event-driven control, which would be triggered only when unexpected events change the evolution of the system. In subsections 1.2 and 1.4, we present a comparison and state of the art on the event-driven control. Our main contributions are presented in Section 3 and represent new probabilistic and analytic methods for computing the energy model of the system. By energy model we refer exactly to the computation of the quadratic expression witch is generally associated with the energy consumed to apply the control and maintain the system inside the control area, summed with the energy of the internal states of the system. We therefore propose new analytical methods for determining the following parameters needed to construct the energy model: (1) the first and second order moment of the state variable, (2) the exit times from the control zone and (3) the control periods.

Although the study of the considered SSI might seem trivial at a first glance, the stochastic behavior of the system induces a high level of uncertainty when applying the event-driven control over random and periodical restart periods. This increases the complexity of the problem and leads us to the probabilistic analytic methods which we further propose. The present paper completes and extends our previous work regarding the exit times and probabilities, which have been published in (Mihăiţă, Mocanu, 2012). We conclude the paper with a numerical example in Section 4, for the case of a two-state controlled stochastic switching integrator, followed by conclusions and further perspectives.

1.2. Event-driven control

The continuous evolution of the stochastic switching systems is interspersed by discrete events, which modify the differential equations defining the continuous flow. The switches between the states of the system produce discontinuities in the trajectories describing the dynamics of the system, leading to weaker solution concepts for the differential equation such as the Filippov solutions (Beek *et al.*, 2004) or the Zeno behavior (Lygeros *et al.*, 2003). This aspect hinders the design of a suitable controller for continuous-time hybrid systems, for example when the computational method involves non-convex problems (Xu, Antsaklis, 2003). In the discrete-time case, the optimal control problems are easier to solve as the switching between the system states is caused by events that take place at certain fixed sampling instants (Bemporad, Morari, 1999). But if the switches occur during the inter-sampling periods, this would lead to modeling errors and state-mismatch.

In order to avoid state-mismatch errors, an appropriate solution would be to apply and change the command values exactly when a switch occurs, in other words to apply an *event-driven control* (EDC). Event-driven methods have been originally used in Petri nets and finite-state machines, and started to gain popularity in industrial systems (Guzzella, Onder, 2006), communication networks (Wang, Lemmon, 2008) and biological systems (Wilson, 1999). The main advantage of using the EDC is the fact that it can be applied only when is needed, becoming very efficient when the control measures or the data acquisition is expensive, or when the energy consumption has certain limitations (Cogill *et al.*, 2007; Åström, Bernahardsson, 2002).

In the literature, the EDC can be found under different notations: event oriented sampling (Åström, 2008), Lebesque sampling (Otanez *et al.*, 2002) or asynchronous command (Heemels *et al.*, 1999). If the first studies (Kwon *et al.*, 1999) have analyzed the EDC only through simulations and experiments, subsequent studies, (Åström, Bernahardsson, 2002) have obtained analytical results for a first order system, by comparing the performance of the event-driven controlled systems and the continuous-sampling controlled systems. They have shown that the EDC provides better performance under certain experimental conditions, caused by the fact that the command can be decreased on time intervals having fewer perturbations, and increased on time intervals with high stochastic perturbations. For example, (Henningsson *et al.*, 2008) have shown that sporadic control can give better performance than periodic control in terms of both reduced process state variance and reduced control action frequency. More recent studies (Stöcker *et al.*, 2013) focus on the decentralized event-driven feedback control of physically interconnected systems, and show how the number of events can be reduced by estimating the interconnection signals between the subsystems.

In Section 2 of the paper we present the application of the event-driven control over the stochastic switching system we study, in order to maintain the system evolution inside a specific control zone.

1.3. Hybrid automaton

The basis for a hybrid system modeling framework is often provided by a *hybrid automaton*, as the discrete events cause transitions from one discrete state to another, regardless whether they are controlled or uncontrolled. When found in a particular state, the system's behavior can be described by differential equations. Moreover, when the stochastic setting appears, one might be interested in the construction of stochastic hybrid automatons (Cassandras, 2008), or in the use of piecewise-deterministic Markov processes (Schäl, 1998). Due to their dual behavior (discrete and continuous) and to the abrupt variations in their structure, the Markov chains are ideal to analyze and represent the stochastic switching systems (Mao, Yuan, 2006; Boukas, 2006). In this article we use the continuous-time Markov Chains (CTMC) to model the system behavior and the stochastic transitions between the states.

1.4. Alternative approaches

Computing the performance of a stochastic hybrid system is generally a hard task due to the lack of a closed form expression which would capture, for example, the dependence between some performance metrics and control parameters (Yao, Cassandras, 2011). In the domain of pure discrete event systems (DES), the infinitesimal perturbation analysis (IPA) has gained popularity as it offers the possibility to obtain unbiased estimates of performance metric gradients, which can be later incorporated intro gradient-based algorithms for optimization purposes (Panayiotou, Cassandras, 2006). However, when dealing with DES presenting discontinuities in their behavior, the IPA estimates become biased, and therefore unreliable for applying the control. Recent studies have shown that the IPA framework can be applied on a particular class of stochastic hybrid systems, more specifically, on Stochastic Flow Models (SFMs). For example, (Wardi, Riley, 2010; Cassandras *et al.*, 2010), generalize and optimize the application of the IPA framework.

Another alternative of applying the control over stochastic hybrid systems, is the predictive optimal stochastic control, which takes into account the probabilistic uncertainty in dynamic systems, and aims to control the predicted distribution of the system state in an optimal manner. (Blackmore *et al.*, 2007) worked on an approximation method of the stochastic predictive control problem to a deterministic one, and solved the optimization problem using Mixed Integer Linear Programming. Recently, a numerical approximation of stochastic hybrid processes with jumps by locally consistent Markov decision processes can be found in (Temoçin, Weber, 2014). Other works address the optimal control problem of Markov chains with constraints (Miller *et al.*, 2010) or use optimal control techniques to obtain the stabilization of stochastic switched systems (Corona *et al.*, 2014).

Whilst in the current paper we do not address the optimal control problem of stochastic switching systems, the present work can be considered as a first step towards the application of optimal control methods. res

2. Stochastic switching integrator (SSI)

2.1. Uncontrolled SSI

Let $s \in S$ (a countable set) denote the discrete state, and $x \in X \subseteq \mathbb{R}$ denote the continuous state, or more explicitly, the state variable. A sample path of such a system would consist of a sequence of intervals of continuous evolution, followed by discrete transitions. The system remains in a discrete state s until a random switch will occur. We define the system by the following equation:

$$\begin{cases} \dot{x}(t) = r_{Z(t)} \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x_0 \in \mathbb{R}$ is the initial state of the system, $Z(t)$ is the Markov chain defined on the finite state space $S = \{1, 2, \ldots, N\}$, and $r_{Z(t)}$ are non-zero switching values, which are chosen so that:

$$\begin{cases} r_i > 0, \forall i \in \{1, \ldots, M\} \\ r_j < 0, \forall j \in \{M+1, \ldots, N\} \end{cases}$$

More explicitly, the system has half of the states characterized by positive switching rates, while the other half is characterized by negative switching rates. Dividing the states of the SSI in two halves is assumed for simplicity, without loss of generality. The further proposed analytical methods can be applied independently of the proportion of the switching rates associated to the states.

The hybrid behavior of the systems is characterized by the continuous evolution of the state variable $x(t)$, and the discrete switches between the states of the associated Continuous-time Markov Chain. We also consider the corresponding transition rate matrix:

$$\mathbf{Q} = \begin{pmatrix} -\sum\limits_{j\neq 1} \lambda_{1,j} & \ldots & \lambda_{1,N} \\ & \ddots & \\ \lambda_{N,1} & \ldots & -\sum\limits_{j\neq N} \lambda_{N,j} \end{pmatrix} \tag{2}$$

where $\lambda_{i,j}$ is the transition rate from state *i* to state *j*. Consequently, the *transition probability between* i *and* j is $p_{i,j} = \lambda_{i,j} / \sum_{j \neq i} \lambda_{i,j}$.

A graphical representation of a two-state stochastic switching integrator is given in Figure 1, with the associated transition rates: $\{r_1 > 0\}$ and $\{r_2 < 0\}$. Using the notations from (Cassandras, 2008), we consider σ_1 and σ_2 to be the events triggering the transitions from state 1 to state 2, respectively from state 2 to state 1.

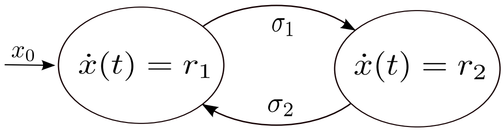

Figure 1. Two-state stochastic switching system

2.2. Controlled SSI

Let us consider X_{min} and X_{max} the minimal, respectively the maximal *control limit*, as shown in Figure 2, and X_L, X_H the *low*, respectively the *high stopping control boundary*.

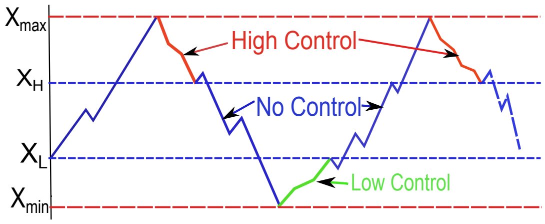

Figure 2. EDC applied over the Stochastic Switching Integrator (1).

The first objective when applying the EDC is to maintain the state variable $x(t)$ inside the *control zone* $[X_{min}, X_{max}]$ with a minimal control energy. Every time one of the limits is reached (for example when $\{x(t) = X_{min}\}$ or $\{x(t) = X_{max}\}$), the control will be applied until $x(t)$ returns in the *no-control zone* (NC). By no-control zone we denote either the interval (X_L, X_H), or $(X_{min}, X_L) \cup (X_H, X_{max})$, as long as the limits haven't been reached and the EDC is not being applied.

In this paper we follow a control scheme which is based on the discrete events of a stochastic process. Therefore, the control process and the controller are stochastic as well. As the controller we build is not an integrator, we do not currently have a fixed set-point (our objective is to maintain $x(t) \in [X_{min}, X_{max}]$), and therefore we do not have an error saturation and integration problem. While other control techniques can be imagined for the specified problem (anti-windup, PID, Lyapunov, etc.), our main objective remains to propose a control scheme based on a stochastic process, which will be further used for the optimal control of other stochastic processes. The current method is therefore inspired from state-feedback control in a simplified stochastic version. We would like to make the observation that indeed, the original approach of the event-driven control formulated by (Åström, Bernahardsson, 2002) had similar characteristics with the relay control (triggered when exceeding a limit) and was our inspirational source for the current study.

We will further use the notation of *high control* (HC) when the EDC is applied in the *upper control zone* $(X_H, X_{max}]$ (this requires that the control has been triggered by X_{max}), respectively *low control* (LC) when EDC is applied in the *lower control zone* $[X_{min}, X_L)$. If the control has not been triggered, the system will rest in the no control zone (NC) (see Figure 2). The controlled switching integrator now becomes:

$$\begin{cases} \dot{x}(t) = r_{Z(t)} + u_{Z(t)}(x(t)) \\ x(0) = x_0 \end{cases} \tag{3}$$

where $u_{Z(t)}(x(t))$ is the control command applied on the system when being in a specific controlled state $\{i, j\} \in S$, of the Markov chain $Z(t)$:

$$u_{Z(t)}(x(t)) = \begin{cases} 0, & \text{if } x(t) \in NC \text{ zone} \\ -QH_i, & \text{if } x(t) \in HC \text{ zone}, \forall i \in S \\ +QL_j, & \text{if } x(t) \in LC \text{ zone}, \forall j \in S \end{cases} \tag{4}$$

and QH_i and QL_j are the *high* and the *low control* applied when the Markov chain is in state i, respectively in state j.

In order for the event-driven control to take place and for the system to return in the no-control zone, QH_i and QL_j are chosen so that:

$$\begin{cases} r_i - QH_i < 0 & , \forall QH_i > 0, i \in S \\ r_j + QL_j > 0 & , \forall QL_j > 0, j \in S \end{cases} \tag{5}$$

An an example, in Figure 3 we represent a *controlled* two-state switching integrator, as defined in (3), using a stochastic hybrid automata (Perez Castaneda *et al.*, 2009). We denote the state space as: $S = \{1NC,\, 2NC,\, 1HC,\, 2HC,\, 1LC,\, 2LC\}$, where $\{1NC, 2NC\}$ are the system states without control, $\{1HC, 2HC\}$ are the states with high control applied inside $[X_H, X_{max}]$, while $\{1LC, 2LC\}$ are the states with low control applied inside $[X_{min}, X_L]$. For example, the state $\{1NC\}$ indicates that the Markov chain is in state 1 without control $\{\dot{x}(t) = r_1\}$, unlike the state $\{1HC\}$ which indicates that the system is in state 1 with high control, therefore $\{\dot{x}(t) = r_1 - QH_1\}$.

Using the representation from Figure 3, we notice that the event-driven control is triggered only when certain types of events will appear. We identify two types of events that can change the dynamics of the system:

1. ***uncontrolled Markov chain events***: they are independent of the extreme frontiers, making the system to switch between states having the same type of control (high, low, or none); in the above example these events are represented by $\{\sigma_1, \sigma_2\}$;

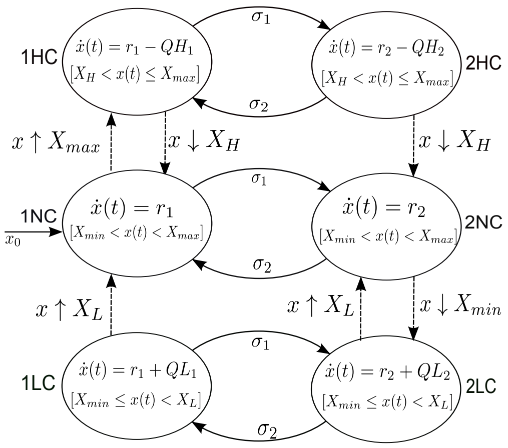

Figure 3. Stochastic hybrid automaton for the controlled two-state SSI [3].

2. ***control events***: they change the type of control applied to the system. For example $\{x \uparrow X_{max}\}$ and $\{x \downarrow X_{min}\}$ are the events which trigger respectively the high and the low control, while $\{x \downarrow X_H\}$ and $\{x \uparrow X_L\}$ represent the events causing the high or low control to stop, when one of the stopping control limits has been reached.

Figure 3 reveals that the system can only reach X_{max} from a state j with a positive associated transition rate ($r_i > 0$). Similarly, the system can only reach X_{min} from a state j having a negative associated transition rate ($r_j < 0$). When $\{x(t) = X_{max}\}$ the high control is applied and the system dynamics becomes $\{\dot{x}(t) = r_{Z(t)} - QH_i < 0\}$, causing the state variable to decrease to $\{x(t) = X_H\}$, condition at which the EDC will be stopped (same for $\{x(t) = X_{min}\}$). At X_H the system will be in one of the corresponding uncontrolled state $\{1NC, 2NC\}$.

3. Energy model

In this paper, we follow the classical Linear Quadratic criterion, which is used for applying an optimal control over linear systems. According to (Anderson, Moore, 1990), linear controllers are achieved by working with quadratic performance indices, which are quadratic in the control and regulation error variables. In general, when we need to select a performance index for a regulator, the cost terms are constructed using the control energy and the energy associated with the internal state of the system. These methods which are meant to achieve linear optimal control are known as Linear-Quadratic (LQ) methods. We therefore propose an LQ-like method for constructing the energy model of the considered stochastic switching system. We remind that by energy model we refer exactly to the computation of the quadratic expression witch is generally associated with the energy consumed to apply the control and maintain the system inside the control area, summed with the energy of the internal states of the system. The analytical methods proposed here can be considered as a first step towards the application of a linear optimal quadratic control measure.

From the previous analysis of the system, we observe different sources of stochastic uncertainty: i) the initial state uncertainty, ii) the initial departing point uncertainty, iii) the random mode transitions, and iv) the random control periods over which we apply the event-driven control method.

As the transitions between the system states are exclusively random, arriving at the borders and applying the EDC is a random process as well. Let's denote by T_r the random period that the system needs to start in X_H or X_L, apply control and return to the no-control zone. The T_r period refers to all the possible restart periods during the evolution of the system, and not only to the one starting at $t = 0$. Over this type of period, we define the following quadratic cost criterion:

$$J_{T_r} = E\left[\int_0^{T_r} [q \cdot x^2(t) + r \cdot u^2(t)]dt\right], q, r > 0. \tag{6}$$

where $u^2(t)$ is the consumed control energy and $E[x^2(t)]$ is the second order moment of the state variable. For the computation of the above cost criterion, we need to determine: T_r, $u^2(t)$ and $E[x^2(t)]$. If one needs to determine the total quadratic cost criterion associated to the whole functioning of the system, then he/she will need to compute the sum of the costs associated to all the restart periods that may occur.

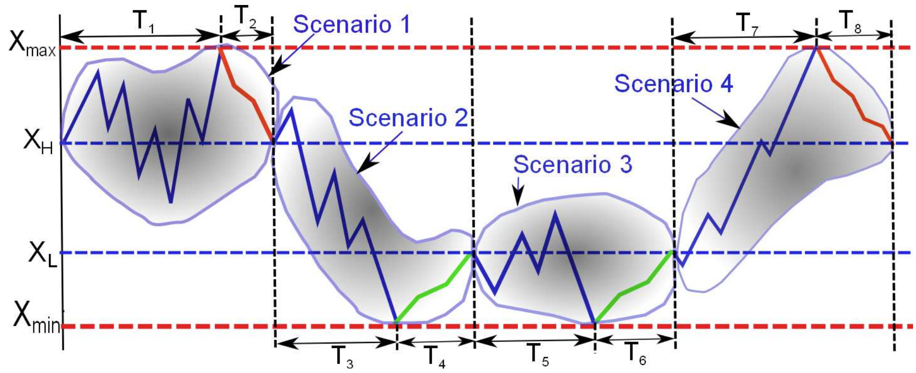

Figure 4. Possible evolution scenarios starting from X_H and X_L.

In the remainder of this section we present the analytical methods for determining each of the above parameters needed to compute J_{T_r}. In Figure 4, we represent a possible trajectory of the system's evolution, for which we identify four possible scenarios of evolution, as the system can either start in:

1. X_H and reach X_{max} during T_1, which triggers the HC during T_2, until the system returns to X_H,

2. X_H and reach X_{min} during T_3, which triggers the LC during T_4, until the system arrives at X_L,

3. X_L and reach X_{min} during T_5, which triggers the LC during T_6, until the system returns to X_L and,

4. X_L and reach X_{max} during T_7, which triggers the HC during T_8, until the system arrives at X_H.

From the periodic behavior of the system, we write the restart period T_r as:

$$T_r = T_{exit} + T_{control} \tag{7}$$

where $T_{exit} \in \{T_1, T_3, T_5, T_7\}$ represents the *exit time from the control zone*, while $T_{control} \in \{T_2, T_4, T_6, T_8\}$ is the time needed to apply the high or the low control. We will often refer to $T_{control}$ as the *the control period*. We make the observation that the T_{exit} times are computed from the last time the system has started either in X_H, or in X_L, and not only from $t = 0$. The analytic methods for computing these times are presented in Sections 3.2 and 3.3. By considering the scenarios (1)-(4), we write the energy consumed for applying the EDC as:

$$En^j_{tot} = p^j_1 E^j_1 + p^j_2 E^j_2 + p^j_3 E^j_3 + p^j_4 E^j_4. \tag{8}$$

where E^j_i, $i \in \{1 \ldots 4\}, j \in S$ is the energy consumed in the i^{th} scenario, when starting from the j state. In the current paper we determine these energies, knowing that the occurrence probabilities of each scenario (p^j_i) have been presented in (Mihăiţă, Mocanu, 2012).

3.1. Second order moment of the state variable

As previously stated, determining the quadratic cost criterion implies computing: i) the restart period T_r, ii) the control energy $u^2(t)$, and iii) the second order moment of the state variable $E[x^2(t)]$.

First we compute $E[x^2(t)]$. The method we present in this section is inspired by the analytical method used by (Viswanadham *et al.*, 1995) for computing the performability moments in the transient analysis of a failure-prone system. We express the n^{th} order moment of the state variable when the system is in state i, as:

$$m_{n,i}(t) = E[x^n(t)|Z(0) = i], \quad i \in \{1, 2, \ldots, N\}$$

and $\vec{m}_n(t) = [m_{n,1}, \ldots, m_{n,N}]$ as the *vector of the state variable moments*, having $\{\vec{m}_0(t) = \vec{e}, t > 0\}$ (the zero performability moment is the identity vector), and $\vec{m}_n(0) = \vec{0}$ as at $t = 0$ all the performability moments are null (Viswanadham *et al.*, 1995).

We write the equation for the first order moment $m_1(t)$, which corresponds to the mean state variable of the stochastic switching integrator from (3):

$$\frac{d\vec{m}_1(t)}{dt} = \mathbf{Q}\vec{m}_1(t) + \mathbf{R}\vec{m}_0(t), \qquad \vec{m}_0(t) = \vec{e}$$

where $\mathbf{Q}$ is the matrix generator of the Markov Chain (2), and $\mathbf{R}$ the diagonal matrix of transition rates:

$$\mathbf{R} = \begin{pmatrix} r_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & r_N \end{pmatrix}$$

By left-multiplying the above equation with e^{-tQ} we obtain:

$$\begin{aligned} e^{-tQ}\frac{d\vec{m}_1(t)}{dt} - e^{-tQ}Q\vec{m}_1(t) &= e^{-tQ}\mathbf{R}\vec{m}_0(t) \Leftrightarrow \\ \frac{d}{dt}[e^{-tQ}\vec{m}_1(t)] &= e^{-tQ}\mathbf{R}\vec{m}_0(t) \Rightarrow \\ \vec{m}_1(t) &= \int_0^t e^{(t-\tau)\mathbf{Q}}\mathbf{R}\vec{e}d\tau \end{aligned} \tag{9}$$

which is used in the calculus of the expected first order moment of the state variable:

$$E[x(t)] = \vec{p}(0)^T\vec{m}_1(t)$$

Following the same procedure for the second order moment $m_2(t)$, we write:

$$\frac{d\vec{m}_2(t)}{dt} = \mathbf{Q}\vec{m}_2(t) + 2\mathbf{R}\vec{m}_1(t) \tag{10}$$

where $\vec{m}_1(t)$ is previously calculated in equation (9). By applying the same method as in the case of $\vec{m}_1(t)$ (which can be adapted for performability moments of higher order), we obtain:

$$\vec{m}_2(t) = 2\int_0^t e^{(t-\tau)\mathbf{Q}}\mathbf{R}\vec{m}_1(\tau)d\tau$$

later used for computing the expected second order moment of the state variable:

$$E[x^2(t)] = \vec{p}(0)^T \cdot \vec{m}_2(t) \tag{11}$$

that we were searching for in (6).

3.2. Computing the exit times from the control zone

In many stochastic applications, predicting the exact time when the system reaches a certain performance level is a hard task due to the random switching in the behavior of the system. The notion of *exit time* (hitting time) is frequently used in finance: determine the right moment to buy or to sell shares (Masoliver *et al.*, 2005), in the manufacturing industry: know when to stop the production if the level of performability has been reached (Rao, Swift, 2006), or in the diffusion process: know when a particle must exit the control zone (Lefebvre, 2011).

The numerical methods for the stochastic differential equations become complicated and inefficient when approximating the exit times (Higham *et al.*, 2011). Certain progress has been made in the work of (Brandejsky *et al.*, 2012), which presents a numerical method (quantification) for computing the exit times for a piecewise deterministic Markov process.

In this section we propose an analytic method for computing the exit time from the control zone $[X_{min}, X_{max}]$, which is inspired from the studies of (Gardiner, 2004) for determining the exit times that a particle needs to reach a control zone with absorbing barriers. Let us define the exit times from a general point of view.

DEFINITION 1. — *Let $\{Z_n\}_{n\in\mathbb{N}_+}$ be a Continuous Time Markov Chain and B a subset of the state space taking values in $[X_{min}, X_{max}]$. We define the exit time from B, when the system starts in state j, as:*

$$T^j_{exit}(x) = \inf\{t > 0 | x(t) \notin B, x(0) = x, Z(0) = j\}.$$

Considering that our stochastic switching integrator is a multi-state system characterized by a Continuous-time Markov Chain, we use backward Kolmogorov equations, for computing the exit times towards the control limits. For more information on the use of backward stochastic differential equations, which generalize the Kolmogorov equation associated to jump Markov processes, the reader can refer to (Confortola, Fuhrman, 2013).

We present only the case for the exit towards X_{max}, as the exit towards X_{min} is similar. Let us consider the following backward Kolmogorov equation :

$$\mathbf{R} \cdot \frac{d\vec{\gamma}_{up}(x)}{dx} + \mathbf{Q}^T \cdot \vec{\gamma}_{up}(x) + \vec{\pi}_{up}(x) = 0 \tag{12}$$

where $\vec{\pi}_{up}(x)$ is the exit probability vector calculated in (Mihăiţă, Mocanu, 2012) and $\vec{\gamma}_{up}(x)$ is the column function vector:

$$\vec{\gamma}_{up}(x) \quad = \quad \begin{bmatrix} \gamma^1_{up}(x) & \gamma^2_{up}(x) & \dots & \gamma^N_{up}(x) \end{bmatrix}^T$$

We consider $\gamma^j_{up}(x)$ to be a function of the couple "exit time $T^j_{up}(x)$ - exit probability $\pi^j_{up}(x)$" towards X_{max}, when departing from x in state j. By $T^j_{up}(x)$ we denote the exit time towards X_{max} when departing from x in a state j as previously defined. The same reasoning applies for the $\gamma^j_{dw}(x)$ function towards X_{min}. Therefore, we write $\gamma^j_{up}(x)$ and $\gamma^j_{dw}(x)$ as:

$$\gamma^j_{up}(x) = \pi^j_{up}(x) \cdot T^j_{up}(x) \tag{13}$$

$$\gamma^j_{dw}(x) = \pi^j_{dw}(x) \cdot T^j_{dw}(x) \tag{14}$$

which respect the following boundary conditions:

$$\begin{aligned} \gamma^j_{up}(X_{min}) &= \gamma^j_{dw}(X_{min}) = 0, \text{ if } r_j < 0 \\ \gamma^j_{up}(X_{max}) &= \gamma^j_{dw}(X_{max}) = 0, \text{ if } r_j > 0 \end{aligned}$$

These conditions can be explained as follows: the exit probability towards X_{max} is zero, when departing from X_{min} in a state with a negative associated transition rate (similarly for the exit probability towards X_{min} from X_{max}, in a state having positive transition rate).

If we left-multiply the equation (12) by $e^{\int_{X_{min}}^{x}(\mathbf{R}^{-1}\mathbf{Q}^{T})dx}\mathbf{R}^{-1}$, and use the notation $\mathbf{W} = -\mathbf{R}^{-1}\mathbf{Q}^{T}$, we obtain the following solution:

$$\vec{\gamma}_{up}(x) = e^{\mathbf{W}(x-X_{min})}\vec{\gamma}_{up}(X_{min}) - \int_{X_{min}}^{x} e^{\mathbf{W}(x-\tau)}\mathbf{R}^{-1}\cdot\vec{\pi}_{up}(\tau)d\tau.$$

We proceed in the same way for the lower case, and obtain $\vec{\gamma}_{dw}(x)$ as:

$$\vec{\gamma}_{dw}(x) = e^{\mathbf{W}(x-X_{min})}\vec{\gamma}_{dw}(X_{min}) - \int_{X_{min}}^{x} e^{\mathbf{W}(x-\tau)}\mathbf{R}^{-1}\cdot\vec{\pi}_{dw}(\tau)d\tau.$$

Knowing $\vec{\gamma}_{up}(x)$ and $\vec{\gamma}_{dw}(x)$, one can easily compute from (13)-(14) the exit times we were searching for: $\{T_1^j, T_3^j, T_5^j, T_7^j\}$, associated to a state $j \in S$:

$$T_{up}^{j}(x) = \frac{\gamma_{up}^{j}(x)}{\pi_{up}^{j}(x)}; \qquad T_{dw}^{j}(x) = \frac{\gamma_{dw}^{j}(x)}{\pi_{dw}^{j}(x)} \tag{15}$$

These exit times will be used for the calculation of the restart period T_r, together with the control periods that we present in the next section.

3.3. Control period

In this section we construct an analytical method for computing the control times needed to apply the EDC (T_2, T_4, T_6, T_8 from Figure 4). The analysis of the system's behavior during the application of high EDC, reveals that the high control periods are equal $T_2 = T_8$, as well as the low control periods: $T_4 = T_6$.

As an observation, the control periods can be also considered as exit times, but from the control zone: $[X_H, X_{max}]$ or $[X_{min}, X_L]$. In this case, the exit limit is causing the EDC to stop (X_L or X_H). Therefore, the current analytical method for computing the control period, is constructed by adapting the method from the Section 3.2 to the above specifications. Our main goal here is to determine the total control period, *independently of the starting point, X_L or X_H, or the starting state*. We make the following observations:

1. The $\mathbf{R}$ matrix is now a diagonal transition rate matrix containing $\{r_j - QH_j < 0\}$ for the high control, and $\{r_j + QL_j > 0\}$ for the low control. Let $\mathbf{R}_H$ be the *high control transition rate matrix* in $[X_H, X_{max}]$, where:

$$\mathbf{R}_H = [r_{ij}]_{i,j\in S} \text{ and}$$
$$r_{ij} = \begin{cases} r_i - QH_i & \text{, if } i = j, \forall i, j \in S \\ 0 & \text{, otherwise} \end{cases} \tag{16}$$

and $\mathbf{R}_L$ the *low control transition rate matrix* in $[X_{min}, X_L]$, where:

$$\mathbf{R}_L = [r_{ij}]_{i,j \in S}$$
$$r_{ij} = \begin{cases} r_i + QL_i & \text{, if } i = j, \forall i, j \in S \\ 0 & \text{, otherwise} \end{cases} \tag{17}$$

2. The boundary conditions are now applied on the control intervals. Let $T_e^j(x)$ be the *exit time from the control zone*, which respects the following boundary conditions:

$$T_e^j(X_H) = 0, \text{if } x(t) \in [X_H, X_{max}] \tag{18}$$

$$T_e^j(X_L) = 0, \text{if } x(t) \in [X_{min}, X_L] \tag{19}$$

3. When we have defined the SSI, we considered that the system can reach X_{max} in ***any*** state having a positive transition rate ($r_j > 0$, $\{j \in S\}$), respectively X_{min} in any state having a negative transition rate ($r_j < 0$). The following analytical method takes into account this aspect.

Using the notations from (Gardiner, 2004), let $G_j(x,t)$ be the *cumulative distribution function* for a state j, which corresponds to the probability that the system will exit the control zone in a certain T_e period: $G_j(x,t) = Prob_j(T_e \geq t)$. The exit probability vector then becomes $\vec{G}(x,t) = [G_j(x,t)]_{j \in S}$, which respects the following boundary conditions:

$$\vec{G}(x,0) = \begin{cases} \vec{\pi}_0^T & \text{, if } X_{min} \leq x \leq X_{max} \\ \vec{0} & \text{, otherwise.} \end{cases} \tag{20}$$

where $\vec{\pi}_0^T$ is the transposed stationary probability vector ($\dot{\vec{\pi}}_0 = \vec{\pi}_0 \mathbf{Q}$). This means that at $t = 0$, the system can depart in any of the states of the Markov chain, with certain stationary probabilities. Following the reasoning and notations from (Gardiner, 2004), we express $T_e^j(x)$, the *total time that the system spends in a state j before exiting the control zone*, as:

$$T_e^j(x) = \int_0^\infty G_j(x,t)dt.$$

Hereafter, we present only the method for the high control time, inside $[X_H, X_{max}]$, as the low control case is similar. Returning to the studies of (Viswanadham *et al.*, 1995), the distribution function $G_j(x,t)$ satisfies the following partial differential equation:

$$\frac{\partial \vec{G}(x,t)}{\partial t} = \mathbf{R}_H \frac{\partial \vec{G}(x,t)}{\partial x} + \mathbf{Q}^T \vec{G}(x,t) \tag{21}$$

$$\vec{G}(0,t) = \vec{0}, \quad t \geq 0 \tag{22}$$

As $\vec{G}(x,0)$ respects (20), we integrate the equation (21) on $(0,\infty)$, and we obtain the following differential equation:

$$\mathbf{R}_H \frac{d\vec{T}_e(x)}{dx} + \mathbf{Q}^T \vec{T}_e(x) + \vec{\pi}_0 = 0. \tag{23}$$

where $\vec{T}_e(x)$ is the column vector:

$$\vec{T}_e(x) = [T_e^1(x) \quad T_e^2(x) \quad \dots \quad T_e^N(x)]^T$$

and $T_e^j(x)$ represents the time that the Markov Chain stays in a state j, before exiting the control zone, *independently* of the departing state. We therefore call $T_e^j(x)$ the *exit time* which satisfies the boundary condition we have expressed in (18)-(19).

In order to respect our third observation, we adapt the initial probability vector to the requirements of the system. We define the *exit probability vector towards* X_H as:

$$\vec{\pi}_H = \vec{\pi}_0(T_{up}^j(X_H))$$

where $T_{up}^j(X_H)$ is the solution from (15). By normalizing this vector, we obtain the equivalent of the initial probability vector $\vec{\pi}_0$, which we use in (23):

$$\vec{\pi}_{O_H} = [\pi_{O_H}(1) \quad \pi_{O_H}(2) \quad \dots \quad \pi_{O_H}(N)]^T \text{, where}$$

$$\vec{\pi}_{O_H}(j) = \begin{cases} \frac{\vec{\pi}_H(j)}{\sum\limits_{k \in S_H} \vec{\pi}_H(k)}, \text{ if } r_j > 0 \text{ and } S_H = \{k | r_k > 0\} \\ 0, \text{ otherwise.} \end{cases} \tag{24}$$

More explicitly, $\vec{\pi}_{O_H}$ is the normalized restriction of the state subset, having positive transition rates (S_H). Equation (21) now becomes:

$$\mathbf{R_s} \frac{d\vec{T}_e(x)}{dx} + \mathbf{Q}^T \vec{T}_e(x) + \vec{\pi}_{O_H} = 0. \tag{25}$$

which can be solved similarly to the equation (12), using the boundary conditions from (18):

$$\vec{T}_e(x) = e^{\mathbf{W}(x - X_H)} \vec{T}_e(X_H) - \int_{X_H}^{x} e^{\mathbf{W}(x-\tau)} \mathbf{R}_H^{-1} \vec{\pi}_{O_H} d\tau \tag{26}$$

The sum of all the $T_e^j(x)$ gives the *total high control period* that we are searching for in this section (or the total time that the systems spends in the high control states when applying the HC), and which corresponds to the T_2 time from Figure 4:

$$T_2(x) = T_8(x) = \sum_{j=1}^{N} T_e^k(x) \tag{27}$$

A similar method can be implemented to obtain the low control periods: $T_4 = T_6$. This section completes the calculation of the restart period (T_r). Therefore, the exit times from the control zone, as well as the control period can now be computed.

3.4. Control energy

The last parameter still to be determined in equation (6) is the energy consumed for applying the high or the low control. In the previous sections, we computed all the parameters needed to express the control energy as:

$$En^j_{tot} = p^j_1 \frac{e^j_1}{T^j_1 + T^j_u} + p^j_2 \frac{e^j_2}{T^j_3 + T_d} + p^j_3 \frac{e^j_3}{T^j_5 + T_d} + p^j_4 \frac{e^j_4}{T^j_7 + T^j_u}$$

where $\{e^j_1, e^j_2, e^j_3, e^j_4\}$ are the control energies of each scenario, for a state j:

$$e^j_1 = e^j_4 = \int_0^{T^j_u} QH^2_j dt = QH^2_j \cdot T^j_u$$

$$e^j_2 = e^j_3 = \int_0^{T_d} QL^2_j dt = QL^2_j \cdot T_d$$

T^j_u is the high control period that we obtained in (26), and T^j_d is the low control period that can be achieved in a similar manner. We now express the total control energy of the system during T_r, by also taking into consideration the stationary probability vector $\vec{\pi}_0$:

$$En_{tot} = \sum_{j=1}^{N} \pi_j En^j_{tot}. \tag{28}$$

This completes the analytical computation of the three main parameters involved in the calculation of the quadratic cost criterion: T_r, $u^2(t)$ and $E[x^2(t)]$ (6). We notice the complexity of the quadratic cost J_{T_r}, as many control parameters, stopping control boundaries and exit times have specific constraints that need to be respected.

4. Numerical example

In this section, we give a numerical example of the event-driven method applied over a two-state stochastic switching integrator, for which we apply the analytic methods presented above. These analytical results have been used to conceive an event-driven simulation algorithm, proposed in (Mihăiţă, Mocanu, 2011).

Let us consider a two-state SSI having the following parameters: $r_1 = 7$, $r_2 = -4$, $\lambda = 9$, $\mu = 5$, $X_{max} = 1$, $X_{min} = -1$, $X_L = -0.7$, $X_H = 0.7$. The quadratic cost J_{T_r} (6) can be expressed as:

$$J_{T_r} = q \cdot \left(\int_0^{T_r} E\left[x^2(t)\right] dt \right) + r \cdot E\left(\int_0^{T_r} [u^2(t)] dt \right)$$

where the first part of the equation relates to the second order moment of the state variable (11), while the second part to the total control energy which has been consumed when applying the EDC during the T_r restart period (28).

The results for the simulated and analytical energies, as well as for the performability moments using certain fixed control measures and stopping limits can be seen in Table 1. The small errors we have obtained encourages a future study and extension of the analytic methods proposed in this paper for higher order systems.

Table 1. Quadratic costs for the two-state SSI.

Type of measure	Simulation	Analytic	Errors[%]
2^{nd}performability moment	0.1807	0.1809	0.07
High control energy	8.7253	8.7339	0.09
Low control energy	5.1337	51629	0.56
Quadratic Cost	66.6172	65.5179	1.65

Furthermore, one would also want to determine the optimal control values that need to be applied in order to have a minimal consumed energy over the chosen control interval. We can therefore write the minimization problem for our stochastic switching integrator as:

$$\min_{u_{Z(t)}(x(t))} \left[q \cdot \left(\int_0^{T_r} \vec{p}(0)^T \cdot \vec{m}_2(t)dt \right) + r \cdot \left(\sum_{j=1}^{N} \pi_j En_{tot}^j \right) \right], \text{with}$$

$$\begin{aligned} \dot{x}(t) &= r_{Z(t)} + u_{Z(t)}(x(t)) \\ x(t) &\in [X_{min}, X_{max}] \\ r_{Z(t)} + u_{Z(t)}(x(t)) &< 0, \text{ if } x(t) = X_{max} \\ r_{Z(t)} + u_{Z(t)}(x(t)) &> 0, \text{ if } x(t) = X_{min} \\ u_{Z(t)}(x(t)) &\in (0, \infty) \end{aligned}$$

When solving the minimization problem numerically using Matlab or Maple, we obtain the optimal control parameters which can be used to apply the EDC: $QH_1 = 8$, $QH_2 = 0.5817$, $QL_1 = 0.8193$, $QL_2 = 5$, as well as the optimal stopping control boundaries: $X_L = -0.9$, $X_H = 0.9$. In this case the optimal quadratic cost becomes: $J = 64.83$.

A graphical representation of the quadratic cost criterion, when all the above parameters ($X_L, X_H, QH_1, QH_2, QL_1, QL_2$) are varying, can be seen in Figure 5. The number of variations on X_L and X_H has the following signification: for example, the first variation is represented by the pair: $\{X_H = 0.9, X_L = -0.1\}$, while the second variation by $\{X_H = 0.9, X_L = -0.2\}$, etc. In the same way, the first variation for the control parameters is represented by the pair: $\{QH_1 = 7.1, QL_2 = 4.1\}$, while the second one by $\{QH_1 = 7.1, QL_2 = 4.2\}$, and so on, until $\{QH_1 = 8, QL_2 = 5\}$. Further variations on QH and QL can also be tested for this system, but, our experiments show that the optimal QH and QL control parameters stop around these values.

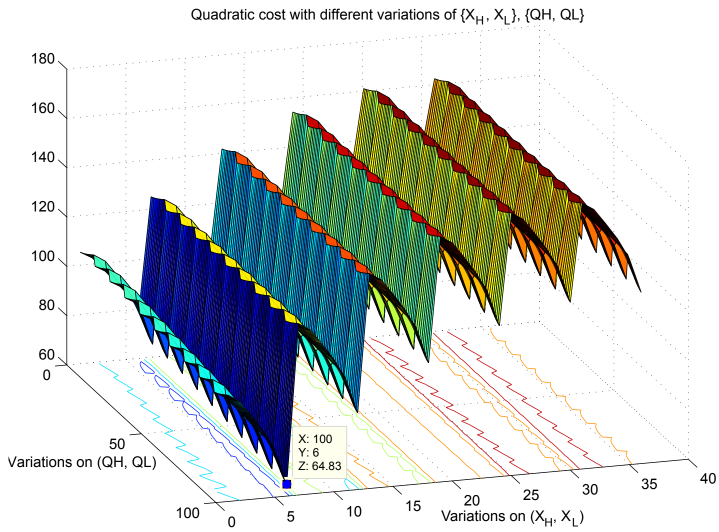

Figure 5. Quadratic cost for different parameter variations.

The optimal values obtained through our numerical experiences lead to the following observation: the control measures QH_1 and QL_2 take bigger values than QH_2 and QL_1, because the control is triggered when arriving at the limits from a state having positive, respectively negative transition rates (no upper constraints have been imposed). The EDC is therefore adapted to the transition rate of each state in order to quickly re-establish the behavior of the system and bring it back in the no-control zone.

We have also tested the impact of the input parameters of a two-state SSI especially on the exit times from the control zone. As the control interval increases, one would expect the exit times to increase as well, and therefore affect the event-driven behavior of the system. The results presented in Annex A validate the analytical exit time method for five other 2-state SSIs with different input parameters, and show accurate results regardless of the values of the switching rates or the control interval.

4.1. Multi-states SSI

While the previous example is meant to facilitate the understanding of the stochastic switching systems we consider in this paper, we conduct as well various experiments for testing multiple-state SSIs, which can be more appropriate when modelling real life systems. The number of parameters to be computed increase with the number of states, therefore increasing the computational complexity and requiring a more detailed analysis. Let us consider a 4-state SSI described by the following input parameters: $r_1 = 9$, $r_2 = -4$, $r_3 = 6$, $r_4 = -8$, $[X_{min}, X_{max}] = [0, 1]$, $[X_L, X_H] = [0.4, 0.8]$ and the corresponding transition rate matrix:

$$\mathbf{Q} = \begin{bmatrix} -45 & 16 & 12 & 17 \\ 9 & -20 & 5 & 6 \\ 10 & 12 & -30 & 8 \\ 12 & 13 & 15 & -40 \end{bmatrix} \tag{29}$$

For a proper analysis comparison, we conducted $N_r = 100.000$ replications of the simulation algorithms, following the four evolution scenarios depicted in Figure 4 (starting randomly in X_L or X_H, reaching the control limits and applying the event-driven control). Analysing final results implies various intermediary output parameters to be validated as well. The detailed analysis and validation of the exit times, control times and scenario control energies is given in Annex B. Table 2 presents the final results for the quadratic cost of the 4-state SSI we consider for this example. Although the control energy errors have increased, we still obtain reasonable errors for the quadratic cost validation (2.13).

Table 2. Quadratic costs for the 4-state SSI.

Type of measure	Simulation	Analytic	Errors[%]
2^{nd}performability moment	1.3890	1.3952	0.44
Total control energy	8.0282	7.9117	1.47
Quadratic Cost	193.3807	189.3401	2.13

Limitations: The main limitation when validating the current method for higher order systems is related to numerical problems that might appear when computing the control periods in Matlab/Latex. The exit times provide accurate results when computed for multiple state SSI, but the singular transition rate matrix which appears in the analytical solution of the control times might affect the numerical computation of the control energies. We have noticed that this kind of errors may appear in general when the control zone is bigger than the uncontrolled zone (which means the system is controlled almost all the time), and is not directly related to the number of states of the SSI.

As an observation, if we compare the current system with queueing models, then the discrete states of our model correspond to the state of the "variable service rate server" in a queue. The state of a queue (number of clients) is here modelled by the continuous state variable $x(t)$, and we do not assume initial constraints on the system state which could cause scalability issues. For a good review on modelling discrete queues with switched linear stochastic models, the user can refer to (Dallery, Gershwin, 1992).

5. Conclusions

In this paper we have presented a probabilistic method for computing the consumed energy when the event driven control is applied over a stochastic switching integrator. The stochastic behavior of the system is represented by the random switches between the states, or the sudden events that trigger the EDC. This aspect complicates the application of the control command and leads us to the construction of the probabilistic energetic model, which can be determined using various parameters such as: the first and second performability moment, the exit times and probabilities, as well as the control periods. The quadratic cost criterion using the above parameters can be further used in the conception of optimal event-driven control techniques. The above analytical methods have been validated for the case of a two and four state stochastic switching integrator.

Perspectives: We observe that for certain system parameters, numerical problems might be encountered due to the singular transition rate matrix appearing in the analytical solutions (26). This aspect motivates our future work to improve the calculation method with respect to the system requirements. We further concentrate on approximation methods for the optimal control, using for example the division of the working space in sub-spaces, depending on the sign of the transition rates which are associated to each state of the system.

We make the observation that in this current study we assume we have full knowledge of the current state of the Markov Chain, and therefore trigger the associated event-driven control. Using Continuous-time Markov Chains indicates that we know the current state of a system which only depends on the previous state and transition, and not on the historical behaviour of the system. Considering uncertainties in the knowledge of such a state can be seen as a future study of our current research work and we could imagine, for example, building a state observer or using Hidden Markov Chain for the considered SSI. This would require further analysis and integration with our current observations and could make the subject of a new research article.

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Annex A

In order to test the impact of the input parameters of the system on the exit times from the control zone, we have conducted various tests on multiple two-state SSIs, as presented in Table 3. We have tested larger switching rates values associated with the states of the model (see Experiment 5 from Table 3), or even larger control intervals (Experiment 4) which would assume longer exit times, therefore multiple switching events between the states.

Table 4 shows the results we obtain for the above experiments, when computing the total exit time taken for the system to exit the control interval $[X_{min}, X_{max}]$. As the control times follow a similar behaviour when triggered, we consider more important to show the results of the total exit times from the control zone. For simplification,

Table 3. Various input parameters for 2-state SSI experiments.

Nr.	r_1	r_2	λ_{12}	λ_{21}	X_{max}	X_{min}	X_H	X_L
1	8	-5	0.7	0.4	60	-60	25	-25
2	10	-15	0.4	0.8	100	-100	50	-50
3	5	-6	0.6	0.7	200	0	150	50
4	20	-20	0.5	0.5	500	-500	200	-200
5	100	-150	0.7	0.3	300	-300	100	-100

we denote, for example, $T_{tot}(X_L)$ as the total exit time needed to exit either X_{min} or X_{max}, when departing in X_L:

$$T_{tot}(X_L) = \Sigma_{i\in S}[T_{up}^i(X_L) + T_{dw}^i(X_L)]$$

Table 4. Total exit time error comparison for experiments in Table 3.

Nr.	Analytic $T_{tot}(X_H)$	Analytic $T_{tot}(X_L)$	Simulation $T_{tot}(X_H)$	Simulation $T_{tot}(X_L)$	Error[%] $T_{tot}(X_H)$	Error[%] $T_{tot}(X_L)$
1	53.97	45.57	53.83	45.35	0.26	0.48
2	188.6	170.3	188.20	170.11	0.21	0.11
3	25.69	41.64	25.65	41.85	0.15	0.50
4	287.50	287.50	290.89	284.31	1.16	1.10
5	12.43	15.55	12.57	15.34	1.11	1.35

Once again, the results confirm the validity of the approach for computing the mean exit times, for various 2-state SSIs. The errors remain inferior to 1.35%, even when the system has larger switching rates (experiment 5) or larger control zones (experiment 4).

Annex B

Firstly, we present the exit times from the control zone (T_{up}^i and T_{dw}^i, $i \in S = \{1, ..4\}$), which we obtain for the 4-state SSI considered in Section 4.1, when applying the analytical methods from Section 3.2. Table 5, presents the numerical results of all the possible exit times obtained through the simulation and the current analytic method. We recall that, for example, $T_{up}^1(X_L)$ stands or the exit time towards X_{max}, when departing from X_L in state 1. The small errors (inferior to 0.62%) validate the current analytical method for the exit time.

Secondly, when computing the control times for the considered 4-state SSI, we obtain the errors presented in Table 6. These control time errors remain inferior to 1.12% and indicate good computational precision when further computing the energies consumed to apply the event-driven control.

Lastly, in Table 7 we present the scenario control energies we have obtained for the 4-state SSI (detailed in Section 3.4). Although each individual control energy presents higher errors when compared to the simulation results, the overall control

Table 5. Exit times towards X_{max}, X_{min} for a 4-State SSI.

Exit times	Simulation	Analytic	Errors[%]
$T_{up}^1(X_L)$	0.2082	0.2085	0.14
$T_{up}^2(X_L)$	0.2683	0.2697	0.51
$T_{up}^3(X_L)$	0.2156	0.2159	0.13
$T_{up}^4(X_L)$	0.2595	0.2592	0.11
$T_{up}^1(X_H)$	0.0840	0.0844	0.47
$T_{up}^2(X_H)$	0.1893	0.1905	0.62
$T_{up}^3(X_H)$	0.0879	0.0874	0.56
$T_{up}^4(X_H)$	0.1767	0.1772	0.28
$T_{dw}^1(X_L)$	0.2482	0.248	0.08
$T_{dw}^2(X_L)$	0.1863	0.1867	0.21
$T_{dw}^3(X_L)$	0.2602	0.2588	0.53
$T_{dw}^4(X_L)$	0.1661	0.166	0.06
$T_{dw}^1(X_H)$	0.3211	0.321	0.03
$T_{dw}^2(X_H)$	0.3083	0.3075	0.25
$T_{dw}^3(X_H)$	0.3278	0.3276	0.06
$T_{dw}^4(X_H)$	0.2833	0.2826	0.24

Table 6. High and low control times for the considered 4-state SSI.

Control time	Simulation	Analytic	Error[%]
$T_2 = T_8$(High Control)	0.0819	0.0828	1.08
$T_4 = T_6$(Low Control)	0.1409	0.1425	1.12

energy remains in acceptable errors, as it is computed using the scenario probabilities and exit times from the control zone, which present accurate results, as seen from previous examples.

Table 7. Control energies for a 4-state SSI.

Control energy	Simulation	Analytic	Error [%]
e_1^1	2.1607	2.1552	4.77
e_1^2	0.0047	0.0046	0.40
e_1^3	1.5068	1.603	6.00
e_1^4	0.0008	0.0007	1.25
e_2^1	0.0043	0.0043	0.00
e_2^2	1.7243	1.8667	7.62
e_2^3	0.0009	0.0008	3.33
e_2^4	2.6413	2.3785	9.94
En_{tot}	8.0282	7.9117	1.47